QUASI-ANOSOV DIFFEOMORPHISMS OF 3-MANIFOLDS

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ABSTRACT. In 1969, Hirsch posed the following problem: given a diffeomorphism $f : N \to N$, and a compact invariant hyperbolic set Λ of f, describe the topology of Λ and the dynamics of f restricted to Λ. We solve the problem where $\Lambda = M^3$ is a closed 3-manifold: if M^3 is orientable, then it is a connected sum of tori and handles; otherwise it is a connected sum of tori and handles quotiented by involutions.

The dynamics of the diffeomorphisms restricted to M^3, called *quasi-Anosov diffeomorphisms*, is also classified: it is the connected sum of DA-diffeomorphisms, quotiented by commuting involutions.

1. INTRODUCTION

This paper deals with hyperbolic sub-dynamics. It is related to a problem posed by M. Hirsch, around 1969: given a diffeomorphism $f : N \to N$, and a compact invariant hyperbolic set Λ of f, describe the topology of Λ and the dynamics of f restricted to Λ. Hirsch asked, in particular, whether the fact that Λ were a manifold M would imply that the restriction of f to M is an Anosov diffeomorphism [11]. However, in 1976, Franks and Robinson gave an example of a non-Anosov hyperbolic sub-dynamics in the connected sum of two $\mathbb{T}^3$ [3] (see below). There are also examples of hyperbolic sub-dynamics in non-orientable 3-manifolds, for instance, the example of Zhuzhoma and Medvedev [18]. We show here that all examples of 3-manifolds that are hyperbolic invariant sets are, in fact, finite connected sums of the examples above and handles $S^2 \times S^1$ (see definitions in §3 and §5)

Theorem 1.1. *Let $f : N \to N$ be a diffeomorphism, and let $M \subset N$ be a hyperbolic invariant set for f such that M is a closed orientable 3-manifold. Then the Kneser-Milnor prime decomposition of M is*

$$M = T_1 \# \ldots \# T_k \# H_1 \# \ldots \# H_r$$

the connected sum of $k \geq 1$ tori $T_i = \mathbb{T}^3$ and $r \geq 0$ handles $H_j = S^2 \times S^1$. In case M is non-orientable, then M decomposes as

$$M = \tilde{T}_1 \# \ldots \# \tilde{T}_k \# H_1 \# \ldots \# H_r$$

the connected sum of $k \geq 1$ tori quotiented by involutions $\tilde{T}_i = \mathbb{T}^3|\theta_i$ and r handles $H_j = S^2 \times S^1$.

In 1976, Mañé obtained the following characterization [15] (see also Theorem 3.3): $g : M \to M$ is the restriction of another diffeomorphism to a hyperbolic set M that is a closed manifold, if and only if g is **quasi-Anosov**; that is, if it satisfies Axiom A and all intersections of stable and unstable manifolds are quasi-transversal, i.e.:

(1.1) $T_x W^s(x) \cap T_x W^u(x) = \{0\}$ $\forall x \in M$

The Franks-Robinson's example of a non-Anosov quasi-Anosov diffeomorphism is essentially as follows: they consider a hyperbolic linear automorphism of a torus T_1 with only one fixed point, and its inverse in another torus T_2. They produce appropriate deformations on each torus (DA-diffeomorphisms) around their respective fixed points. Then they cut suitable neighborhoods containing these fixed points, and carefully glue together along their boundary so that the stable and unstable foliations intersect quasi-transversally. This is a quasi-Anosov diffeomorphism in the connected sum of T_1 and T_2, and hence $T_1 \# T_2$ is a compact invariant hyperbolic set of some diffeomorphism. The non-orientable example by Medvedev and Zhuzhoma [18] is similar to Franks and Robinson's, but they perform a quotient of each T_i by an involution before gluing them together.

The second part of this work, a classification of the dynamics of quasi-Anosov diffeomorphisms of 3-manifolds, shows that all examples are, in fact, connected sums of the basic examples above:

Theorem 1.2. *Let $g : M \to M$ be a quasi-Anosov diffeomorphism of a closed 3-manifold M. Then*

(1) *The non-wandering set $\Omega(g)$ of g consists of a finite number of codimension-one expanding attractors, codimension-one shrinking repellers and hyperbolic periodic points.*

(2) *For each attractor Λ in $\Omega(g)$, there exist a hyperbolic toral automorphism A with stable index one, a finite set Q of A-periodic points, and a linear involution θ of $\mathbb{T}^3$ fixing Q such that the restriction of g to its basin of attraction $W^s(\Lambda)$ is topologically conjugate to a DA-diffeomorphism f_Q^A on the punctured torus $\mathbb{T}^3 - Q$ quotiented by θ. In case M is an orientable manifold, θ is the identity map. An analogous result holds for the repellers of $\Omega(g)$.*

Item (2) above is actually a consequence of item (1), as it was shown by Plykin in [20, 21], see also [6] and [7]. A statement of the result can be found in Theorem 4.3 in this work. The proof of Theorem 1.2 is in §4. Theorem 1.2, in fact, implies Theorem 1.1. This is proved in §5.

Let us see how a handle $S^2 \times S^1$ could appear in the prime decomposition of M: Consider a linear automorphism of a torus T_1, and its inverse in a torus T_2, as in Franks-Robinson's example. Then, instead of exploding a fixed point, one explodes and cuts around an orbit of period 2 in T_1 and in T_2. The rest of the construction is very similar, gluing carefully as in that example to obtain a quasi-Anosov dynamics. This gives the connected sum of two tori and a handle. Explanation and details can be found in §5.

Let us also mention that in a previous work [22] it was shown there exist a codimension-one expanding attractor and a codimension-one shrinking repeller if g is a quasi-Anosov diffeomorphism of a 3-manifold that is not Anosov. The fact that only $\mathbb{T}^3$ can be an invariant subset of any known Anosov system was already shown by A. Zeghib [29]. In that case, the dynamics is Anosov. See also [2] and [16].

This work is also related to a work by Grines and Zhuzhoma [8]. There they prove that if an n-manifold supports a structurally stable diffeomorphism with a codimension-one expanding attractor, then it is homotopy equivalent to $\mathbb{T}^n$, and homeomorphic to $\mathbb{T}^n$ if $n \neq 4$. In a certain sense, the results deal with complementary extreme situations in the Axiom A world: Grines-Zhuzhoma result deals with structurally stable diffeomorphisms, which are Axiom A satisfying the strong transversality condition. This means that all x, y in the non-wandering set satisfy at their points z of intersection: $T_z W^s(x) \pitchfork T_z W^u(y)$. In particular,

$$\dim E_x^s + \dim E_y^u \geq n$$

In our case, we deal with quasi-Anosov diffeomorphisms, which are Axiom A satisfying equality (1.1). In particular, for x, y, z as above, we have $T_z W^s(x) \cap T_z W^u(y) = \{0\}$, so:

$$\dim E_x^s + \dim E_y^u \leq n$$

In the intersection of both situations are, naturally, the Anosov diffeomorphisms.

Observe that it makes sense to get a classification of the dynamical behavior of quasi-Anosov on its non-wandering set, since quasi-Anosov are Ω-stable [15] (see also §3). They form an open set, due to quasi-transversality condition (1.1). Moreover, they are the C^1-interior of expansive diffeomorphisms, that is, they are robustly expansive [14]. However, 3-dimensional quasi-Anosov diffeomorphisms of $M \neq \mathbb{T}^3$ are never structurally stable, so

they are approximated by other quasi-Anosov diffeomorphisms with different dynamical behavior, but similar asymptotic behavior (Proposition 3.2).

Finally, in Section 7 we study quasi-Anosov diffeomorphisms in the presence of partial hyperbolicity (see definitions in §7). We obtain the following result under mild assumptions on dynamical coherence:

Theorem 1.3. *If $f : M^3 \to M^3$ is a quasi-Anosov diffeomorphism that is partially hyperbolic, and either E^{cs} or E^{cu} integrate to a foliation, then f is Anosov.*

2. Basic definitions

Let us recall some basic definitions and facts: Given a diffeomorphism $f : N \to N$, a compact invariant set Λ is a *hyperbolic set* for f if there is a Tf-invariant splitting of TN on Λ:

$$T_xN = E_x^s \oplus E_x^u \qquad \forall x \in \Lambda$$

such that all unit vectors $v^\sigma \in E_\Lambda^\sigma$, with $\sigma = s, u$ satisfy

$$|Tf(x)v^s| < 1 < |Tf(x)v^u|$$

for some suitable Riemannian metric $|.|$. The *non-wandering set* of a diffeomorphism $g : M \to M$ is denoted by $\Omega(g)$ and consists of the points $x \in M$, such that for each neighborhood U of x, the family $\{g^n(U)\}_{n \in \mathbb{Z}}$ is not pairwise disjoint. The diffeomorphism $g : M \to M$ satisfies *Axiom A* if $\Omega(g)$ is a hyperbolic set for g and periodic points are dense in $\Omega(g)$. The *stable manifold* of a point x is the set

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \quad \text{if} \quad n \to \infty\}$$

where $d(.,.)$ is the induced metric; the *unstable manifold* $W^u(x)$ is defined analogously for $n \to -\infty$. If g satisfies Axiom A, then $W^s(x)$ and $W^u(x)$ are immersed manifolds for each $x \in M$ (see for instance [25]). Also, if d_σ is the intrinsic metric of the invariant manifold $W^\sigma(x)$, for $\sigma = s, u$, one has constants $C, \varepsilon > 0$ and $0 < \lambda < 1$ such that, for instance, if $y \in W^s(x)$, and $d_s(x, y) \leq \varepsilon$ for some small $\varepsilon > 0$ then

$$(2.2) \qquad d_s(f^n(x), f^n(y)) \leq C\lambda^n d_s(x, y) \qquad \forall n \geq 0$$

an analogous bound holds for the unstable manifold.

Due to the Spectral Decomposition Theorem of Smale [26], if g is Axiom A, then $\Omega(g)$ can be decomposed into disjoint compact invariant sets, called *basic sets*:

$$\Omega(g) = \Lambda_1 \cup \cdots \cup \Lambda_r,$$

each Λ_i contains a dense orbit. Furthermore, each Λ_i can be decomposed into disjoint compact sets $\Lambda_i = \Lambda_{i,1} \cup \cdots \cup \Lambda_{i,k}$ such that there exists an $n \in \mathbb{N}$ where each $\Lambda_{i,j}$ is invariant and topologically mixing for g^n. A set X is *topologically mixing* for a diffeomorphism f if for each pair of nonempty open sets U and V of X, there is $K > 0$ such that

$$f^k(U) \cap V \neq \emptyset \qquad \forall k \geq K.$$

Note that $\dim E_x^s$ is constant for x varying on a basic set Λ, we shall call this amount the *stable index* of Λ, and will denote it by $\mathrm{st}(\Lambda)$.

For any set $\Lambda \subset M$, let us denote by $W^\sigma(\Lambda)$ the set $\bigcup_{x \in \Lambda} W^\sigma(x)$, where $\sigma = s, u$. We define the following (reflexive) relation among basic sets:

$$\Lambda_1 \to \Lambda_2 \qquad \Longleftrightarrow \qquad W^u(\Lambda_1) \cap W^s(\Lambda_2) \neq \emptyset$$

The relation $\to$ naturally extends to a transitive relation $\succeq$:

$$\Lambda_i \succeq \Lambda_j \qquad \Longleftrightarrow \qquad \Lambda_i \to \Lambda_{k_1} \to \cdots \to \Lambda_{k_r} \to \Lambda_j$$

where $\Lambda_{k_1}, \ldots, \Lambda_{k_r}$ is a finite sequence of basic sets. The diffeomorphism satisfies the *no-cycles condition* if $\succeq$ is anti-symmetric:

$$\Lambda_1 \succeq \Lambda_2 \quad \text{and} \quad \Lambda_2 \succeq \Lambda_1 \qquad \Longrightarrow \qquad \Lambda_1 = \Lambda_2$$

In this case $\succeq$ defines a partial order among basic sets.

We shall call Λ an *attractor* if Λ is a basic set such that $W^u(\Lambda) = \Lambda$. Note that this implies that there exists a neighborhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{N}} f^n(U)$. Similarly, we shall call Λ a *repeller* if Λ is a basic set such that $W^s(\Lambda) = \Lambda$. If g is Axiom A and satisfies the no-cycles condition, then hyperbolic attractors and repellers are, respectively, the minimal and maximal elements of $\succeq$.

A hyperbolic attractor Λ is a *codimension-one expanding attractor* if all $x \in \Lambda$ satisfy $\dim W^u(x) = \dim M - 1$. Codimension-one shrinking repellers are defined analogously.

Note that an attractor can have topological dimension $\dim M - 1$, and still be not expanding. See the survey [9] on expanding attractors for a discussion on this topic.

3. Quasi-Anosov diffeomorphisms

Let $f : N \to N$ be a diffeomorphism of a Riemannian manifold.

Definition 3.1. *The sets $W^s(x)$ and $W^u(x)$ have a point of **quasi-transversal** intersection at x if*

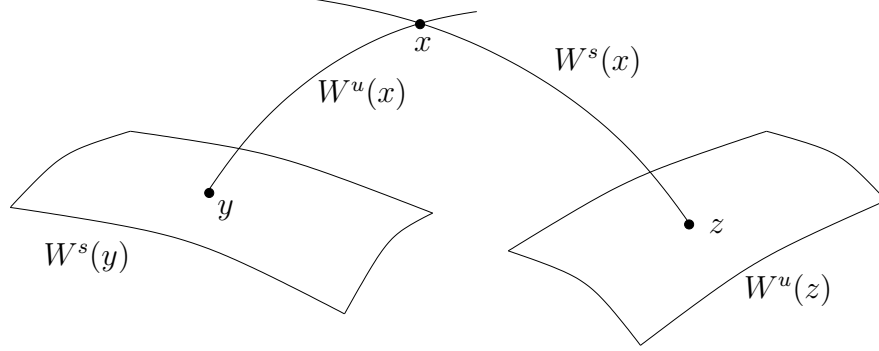$$T_x W^s(x) \cap T_x W^u(x) = \{0\}$$

(see figure 1)

FIGURE 1.   Quasi-transversal intersection at x

At a point of quasi-transversal intersection x, all vectors in E_x^s form a positive angle with vectors in E_x^u. But this does not necessarily imply transversality, as can be seen in Figure 1.

Let us note the difference between this definition and the strong transversality condition. There, transversality is required at the intersection points of $W^u(x)$ and $W^s(y)$, but this can be attained without quasi-transversality, for instance, if we had two planes intersecting at a curve in a 3-dimensional setting.

Observe that a structurally stable quasi-Anosov diffeomorphism is Anosov (see [15] and references therein). On the other hand, quasi-Anosov diffeomorphisms satisfy the no-cycles condition (see below), and hence they are Ω-stable. Also, quasi-Anosov are a C^1-open set od diffeomorphisms[14].

Proposition 3.2. *A quasi-Anosov diffeomorphism f that is not Anosov is approximated by Ω-conjugate quasi-Anosov diffeomorphisms that are not topologically conjugate to f.*

The following theorem by Mañé relates quasi-Anosov diffeomorphisms with hyperbolic sub-dynamics.

Theorem 3.3 (Mañé [15]). *A diffeomorphism g is a quasi-Anosov diffeomorphism if and only if M can be embedded as a hyperbolic set for a diffeomorphism $f : N \to N$ by means of an embedding $i : M \hookrightarrow N$ satisfying $fi = ig$.*

This characterization reduces the proof of Theorem 1.1 to proving Theorem 1.2. See also §5. We shall review some properties of quasi-Anosov diffeomorphisms:

Proposition 3.4. [15] *Quasi-Anosov diffeomorphisms satisfy the no-cycles condition.*

Proof. If Λ_i and Λ_j are two basic sets satisfying $\Lambda_i \to \Lambda_j$, then $W^u(x_i) \cap W^s(x_j) \neq \emptyset$ for some $x_k \in \Lambda_k$. It follows from quasi-transversality that

$$[n - \text{st}(\Lambda_i)] + \text{st}(\Lambda_j) = \dim E^u_{x_i} + \dim E^s_{x_j} \leq n$$

where n is the dimension of M, hence $\text{st}(\Lambda_j) \leq \text{st}(\Lambda_i)$. We get by transitivity that

(3.3)
$$\Lambda_i \succeq \Lambda_j \quad \Rightarrow \quad \text{st}(\Lambda_i) \geq \text{st}(\Lambda_j)$$

Suppose that,

$$\Lambda_1 \to \Lambda_2 \to \cdots \to \Lambda_k \to \Lambda_1.$$

We have, in the first place that $\text{st}(\Lambda_i) = \text{st}(\Lambda_1)$, hence all intersections $x_i \in W^u(\Lambda_i) \cap W^s(\Lambda_{i+1})$ for $i = 1, \ldots, k-1$, and $x_k \in W^u(\Lambda_k) \cap W^s(\Lambda_1)$ are transversal, since

$$[n - \text{st}(\Lambda_i)] + \text{st}(\Lambda_1) = \dim E^u_{x_i} + \dim E^s_{x_1} = n$$

. This implies the x_i's belong to $\Omega(g)$, hence $\Lambda_i = \Lambda_i$ for all $i = 1, \ldots, k$, and so g satisfies the no-cycles condition. $\square$

In the particular case of a quasi-Anosov diffeomorphism of a 3-dimensional manifold, this implies there can be only basic sets with stable index 2 or 1; and basic sets with stable index one can only succeed basic sets with stable index one. We delay the proof of the next proposition until the next section.

Proposition 3.5. *If f is a quasi-Anosov diffeomorphism and Λ_0 is a co-dimension-one expanding attractor, and Λ is a basic set satisfying $\Lambda \succeq \Lambda_0$ with $\text{st}(\Lambda) = 1$, then $\Lambda = \Lambda_0$ or else Λ is a periodic point.*

Analogously, if Λ_0 is a codimension-one repeller, and Λ is a basic set satisfying $\Lambda_0 \succeq \Lambda$ with $\text{st}(\Lambda) = 2$, then $\Lambda = \Lambda_0$ or Λ is a periodic point. This implies:

Proposition 3.6. *All attractors of a quasi-Anosov diffeomorphism of a 3-dimensional manifold are codimension-one expanding attractors, unless the diffeomorphism is Anosov. An analogous statement holds for repellers.*

Proof. Indeed, let Λ_R be a repeller such that $\text{st}(\Lambda_R) = 1$ (hence, not codimension-one). There is a maximal chain of $\succeq$ containing Λ_R. Let Λ_A be a minimal element of that chain. Then, due to (3.3) in the proof above, Λ_A is a codimension-one expanding attractor. But then Proposition 3.5 implies

the repeller Λ_R equals Λ_A, since Λ_A cannot be a periodic point. Therefore $\Lambda_R = \Lambda_A = M$. $\qquad\square$

Note that Propositions 3.5 and 3.6 above prove Item (1) of Theorem 1.2. Item (2) of Theorem 1.2 follows from results in next section.

4. Codimension-one expanding attractors and shrinking repellers - Proof of Theorem 1.2

Before proving Proposition 3.5 we review properties of codimension-one expanding attractors. A codimension-one expanding attractor Λ is *orientable* if the intersection index of $W^s(x) \cap W^u(y)$ is constant at all its intersection points, for $x, y \in \Lambda$. This notion was first introduced by Grines [4, 5]. Let us also recall the following result by Zhuzhoma and Medvedev:

Theorem 4.1 (Medvedev-Zhuzhoma [18]). *If M is an orientable closed 3-manifold, then all codimension-one expanding attractors and shrinking repellers are orientable.*

Derived from Anosov (or DA-) diffeomorphisms were introduced by Smale in [26] (see also [28]). They are certain deformations of hyperbolic automorphisms of the torus. We shall use the following definition [20]:

Corresponding to a hyperbolic toral automorphism A with stable index one, and a finite set Q of A-periodic points, there is a diffeomorphism $f_Q^A : \mathbb{T}^3 \to \mathbb{T}^3$ diffeotopic to A, such that $\Omega(f_Q^A) = \Lambda \cup Q$, where Λ is a codimension-one expanding attractor and Q is a finite set of f_Q^A-repelling periodic points. The stable manifolds of f_Q^A coincide with the stable manifolds of A, except for a finite set of lines $\mathcal{L}_Q$. Each line $L \in \mathcal{L}_Q$ contains a point $q \in Q$. The component of $L - \Lambda$ containing q is an interval whose endpoints $p^{\pm}$ are periodic boundary points of Λ. We call f_Q^A a *DA-diffeomorphism*.

Plykin obtains models for connected codimension-one expanding attractors using DA-diffeomorphisms [20, 21]. See also §8 of [8]. We shall also use some of his intermediate results:

Theorem 4.2 (Plykin [20]). *If Λ is a connected orientable codimension-one expanding attractor of a diffeomorphism $g : M^3 \to M^3$, then $W^s(\Lambda)$ has the homotopy type of $\mathbb{T}^3 - Q$, where Q is a finite set of points. There is a finite point-compactification $\overline{W^s(\Lambda)}$ of $W^s(\Lambda)$ having the homotopy type of $\mathbb{T}^3$, and a homeomorphism $\bar{g} : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ extending $g|_{W^s(\Lambda)}$, and admitting two $\bar{g}$-invariant fibrations that extend, respectively, the stable and unstable manifolds of Λ.*

An analogous result holds for non-orientable attractors: there exists a two-sheeted covering $\pi : \overline{W^s(\Lambda)} \to W^s(\Lambda)$ and a covering homeomorphism

$\bar{g} : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ that commutes with the involution $\theta : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ associated to π, such that $\overline{W^s(\Lambda)}$ has the homotopy type of $\mathbb{T}^3$ [21].

Let us note that results above do not require that Λ has a dense orbit.

Theorem 4.3 (Plykin [20, 21])**.** *If Λ is a connected orientable codimension-one expanding attractor of a diffeomorphism $g : M^n \to M^n$ having a dense unstable manifold, then there exist a hyperbolic toral automorphism A with stable index one, and a finite set Q of A-periodic points, such that $g|_{W^s(\Lambda)}$ is topologically conjugate to the DA-diffeomorphism $f_Q^A|_{\mathbb{T}^n - Q}$.*

If Λ is non-orientable, then there is a two-sheeted covering $\pi : \overline{W^s(\Lambda)} \to W^s(\Lambda)$ with an associate involution $\theta : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$, and a covering homeomorphism $g : \overline{W^s(\Lambda)} \to \overline{W^s(\Lambda)}$ commuting with θ that is topologically conjugate to a DA-diffeomorphism f_Q^A as described above.

Let us note that, in case where the manifold M is a torus, this result was obtained in [7], see also [6] for the two-dimensional case.

Next, we state some of the results obtained in [8] and follow the general outline and notation.

Let Λ be a codimension-one expanding attractor. We will assume for now that Λ is orientable. (The non-orientable case will follow by taking a double cover and looking at the orientable case). A point p is a *boundary point* of a codimension-one expanding attractor Λ if there exists a connected component of $W^s(p) - p$, denoted $W^s_\emptyset(p)$, not intersecting Λ. Boundary points for hyperbolic codimension-one expanding attractors are finite and periodic [20]. For $z \in \Lambda$ and given points $x, y \in W^s(z)$ we denote $(x, y)^s$ (respectively $[x, y]^s$) the open (closed) arc of $W^s(z)$ with endpoints x and y. If p is a boundary point of Λ and $x \in W^u(p) - p$, then there is a unique arc $(x, y)^s_\emptyset$ such that $(x, y)^s \cap \Lambda = \emptyset$ and $y \in \Lambda$. If $z \in W^s(\Lambda) - \Lambda$, then either $z \in (x, y)^s_\emptyset$ for some x and y elements of the unstable manifolds of boundary points, or $z \in W^s_\emptyset(p)$ for some boundary point $p \in \Lambda$.

The boundary points p_1 and p_2 are called *associated* if for each point $x \in W^u(p_1)$ there exists an arc $(x, y)^s_\emptyset$ where $y \in W^u(p_2)$, and similarly for each point $y \in W^u(p_2)$ there is an arc $(x, y)^s_\emptyset$ where $x \in W^u(p_1)$. The boundary point p_1 is said to be *paired* if there exists a boundary point p_2 such that p_1 and p_2 are associated [1]. Two associated boundary points have always the same period m. If $\dim(M) \geq 3$, then all boundary points are paired.

For associated periodic points p_1 and p_2 let

$$\varphi_{p_1,p_2} : (W^u(p_1) - p_1) \cup (W^u(p_2) - p_2) \to (W^u(p_1) - p_1) \cup (W^u(p_2) - p_2)$$

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[1]This concept also appears as *2- bunched* in the bibliography. It is not to be confused with the concept of *center bunching* used for partially hyperbolic diffeomorphisms.

be defined by $\varphi_{p_1,p_2}(x) = y$ whenever $(x,y)^s_\emptyset$. The continuous dependence of stable and unstable manifolds implies that φ_{p_1,p_2} is a homeomorphism. We may naturally extend φ_{p_1,p_2} to be a homeomorphism of $W^u(p_1) \cup W^u(p_2)$ to itself by defining $\varphi_{p_1,p_2}(p_1) = p_2$ and $\varphi_{p_1,p_2}(p_2) = p_1$.

Fix D_{p_1} a closed disk in $W^u(p_1)$ containing p_1 in the interior such that $D_{p_1} \subset \text{int}(f^m(D_{p_1}))$. The boundary of D_{p_1} is a circle denoted S_{p_1}. The circles S_{p_1} and $f^m(S_{p_1})$ bound an annulus contained in $W^u(p_1)$ denoted A_{p_1}.

Since φ_{p_1,p_2} is a homeomorphism we can define

- a closed disk $D_{p_2} = \varphi_{p_1,p_2}(D_{p_1})$ in $W^u(p_2)$,
- a circle $S_{p_2} = \varphi_{p_1,p_2}(S_{p_1})$, and
- an annulus $A_{p_2} = \varphi_{p_1,p_2}(A_{p_1})$.

The set

$$C_{p_1,p_2} = \bigcup_{x \in S_{p_1}} (x, \varphi_{p_1,p_2}(x))^s_\emptyset$$

is called a *connecting cylinder of p_1 and p_2* and is homeomorphic to the open 2-cylinder $S^1 \times (0,1)$. The set

$$S_{p_1,p_2} = D_{p_1} \cup D_{p_2} \cup C_{p_1,p_2}$$

is called a *characteristic sphere for p_1 and p_2* and is homeomorphic to a sphere.

Define

$$A_{p_1,p_2} = \bigcup_{x \in A_{p_1}} [x, \varphi_{p_1,p_2}(x)]^s_\emptyset$$

which is homeomorphic to an annulus times an interval. Let

$$D_{p_1,p_2} = \bigcup_{j \geq 0} f^{jm}(A_{p_1,p_2}) = \bigcup_{x \in W^u(p_1) - \text{int} D_{p_1}} [x, \varphi_{p_1,p_2}(x)]^s_\emptyset$$

and denote π_{p_1} as the projection from D_{p_1,p_2} to $W^u(p_1) - \text{int}(D_{p_1})$. Then the triple $(D_{p_1,p_2}, W^u(p_1) - \text{int}(D_{p_1}), \pi_{p_1})$ is a trivial fiber bundle with fiber the interval $[0,1]$.

The following is Corollary 3.1 in [8].

Lemma 4.4. *Let Λ_0 be a codimension-one orientable expanding attractor and p_1, p_2 are associated boundary points on Λ_0. Suppose Λ is another basic set of M for f with $\text{st}(\Lambda) = 1$. If there exists a point $z \in \Lambda$ such that $W^u(z) \cap D_{p_1,p_2} \neq \emptyset$, then $W^u(z)$ intersects C_{p_1,p_2}.*

Theorem 6.1 in [8] is similar to the following lemma.

Lemma 4.5. *Suppose f is a quasi-Anosov diffeomorphism of a closed 3-manifold M and Λ_0 is an orientable codimension-one expanding attractor of f. Let $\Lambda \neq \Lambda_0$ be a basic set of $\text{st}(\Lambda) = 1$ such that $W^u(\Lambda) \cap D_{p_1,p_2} \neq \emptyset$. Let $C \subset D_{p_1,p_2} \cap W^u(z)$ be a component of the intersection of $D_{p_1,p_2} \cap W^u(z)$,*

where $z \in \Lambda$ is a periodic point. Then $W^u(z) \cap C_{p_1,p_2} = C \cap C_{p_1,p_2} \neq \emptyset$ and this intersection consists of a unique circle, S, that is isotopic to S_{p_1} and S_{p_2}.

We will provide an outline of the proof, (see also [8]), since some of the details are needed in the proof of Proposition 3.5. The statement of Theorem 6.1 in [8] assumes the diffeomorphism is structurally stable. Structurally stable diffeomorphisms have the strong transversality property which implies that for all $x \in \Lambda_0$ and $z \in \Lambda$ that

(4.4)
$$W^s(x) \cap W^u(z) = W^s(x) \pitchfork W^u(z), \text{ and}$$
$$W^u(x) \cap W^s(z) = W^u(x) \pitchfork W^s(z).$$

This is the property used in the proof in [8]. However, if f is quasi-Anosov and Λ_0 and Λ are codimension-one, then for all $x \in \Lambda_0$ and $z \in \Lambda$ property (4.4) holds by the quasi-transversal property of quasi-Anosov diffeomorphisms.

First, we use Lemma 4.4 to show there is a component. Next, we use transversality to show that every component $C \cap C_{p_1,p_2}$ is a circle, S, that is isotopic to S_{p_1} and S_{p_2}. So in fact each component divides C_{p_1,p_2} into two cylinders.

Next, let $B_S \subset W^u(z)$ be a minimal disk bounded by S. Since B_S is minimal it follows that $B_S \cap D_{p_1,p_2} = \emptyset$. Then there are two possibilities.

(1) No B_S contains z.
(2) Some B_S contains z.

It is shown that case (1) can not occur. For case (2) since $f(C_{p_1,p_2}) \subset D_{p_1,p_2}$ we know $f(S) \cap S = \emptyset$ and S is inside $f(S)$ in $W^u(z)$. It then follows that S and $f(S)$ bound a closed annulus in $W^u(z)$ which is a fundamental domain of $W^u(z)$ contained in D_{p_1,p_2}. Thus the intersection of $W^u(z)$ and C_{p_1,p_2} is a unique circle.

Proof of Proposition 3.5. To simplify the argument we first assume the attractors are orientable. Let us suppose that Λ is a basic set that is a codimension-one and Λ is not a hyperbolic attractor. Since periodic points are dense in Λ and Λ_0 we may assume there exist periodic points $x \in \Lambda$ and $x_0 \in \Lambda_0$ such that $W^s(x_0) \cap W^u(x) \neq \emptyset$. Let $y \in W^s(x_0) \cap W^u(x)$. Then from the previous Lemma $y \in (y_1, y_2)^s_\emptyset$ for y_1 and y_2 in the unstable manifolds of associated boundary points p_1 and p_2, respectively.

Let S_{p_1,p_2} be a characteristic sphere for p_1 and p_2 such that $(y_1, y_2)^s_\emptyset \subset C_{p_1,p_2}$, so $C_{p_1,p_2} \cap W^u(x) \neq \emptyset$. From the previous lemma we know that $W^u(x) \cap D_{p_1,p_2}$ is a unique component C. Furthermore, there is a fundamental domain of $W^u(x)$ contained in $D_{p_1,p_2} \subset W^s(\Lambda_0)$. The invariance of $W^s(\Lambda)$ implies that $W^u(x) - x \subset W^s(\Lambda_0)$. Hence, $(W^u(x) - x) \cap \Lambda = \emptyset$. Since $W^u(x) \cap W^s(x)$ is dense in a component of Λ, given by the Spectral

Decomposition Theorem, we know that Λ is trivial and consists of the orbit of x. In this way, the result follows for orientable attractors Λ_0.

We now suppose that Λ is a codimension-one basic set and Λ_0 is a codimension-one *non-orientable* attractor where $\Lambda \to \Lambda_0$. This implies that M is non-orientable from [21]. Let $\bar{M}$ be an orientable manifold and $\pi : \bar{M} \to M$ is a (non-branched) double covering of M. Then there exists a diffeomorphism $\bar{f}$ of $\bar{M}$ that covers f. Furthermore, $\bar{M}$ contains a hyperbolic orientable codimension-one expanding attractor $\bar{\Lambda}_0$ such that $\bar{\Lambda}_0 \subset \pi^{-1}(\Lambda_0)$. The result now follows from the previous argument by lifting Λ. $\square$

5. Proof of Theorem 1.1

Let us recall some basic definitions and results, which can be found in [19]. The **connected sum** of two 3-manifolds is obtained by removing the interior of a 3-cell from each 3-manifold, and then matching the resulting boundaries, using an orientation reversing homeomorphism. The connected sum of M and M' is denoted $M \# M'$. In order to **add a handle** to a connected 3-manifold M, one removes the interior of two disjoint 3-cells from M, and matches the resulting boundaries under an orientation reversing homeomorphism. If one adds a handle to M, one obtains a manifold isomorphic to $M \# S^2 \times S^1$. Note that $M \# S^3 = M$.

A manifold $M \neq S^3$ is **prime** if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$. We have the following Unique Decomposition Theorem (see also [13]):

Theorem 5.1 (Milnor [19]). *Every 3-manifold $M \neq S^3$ can be written as a finite connected sum:*
$$M = M_1 \# \ldots \# M_k$$
where each M_i is prime, $i = 1, \ldots, k$, and is unique up to order and isomorphisms.

Note that the handles $S^2 \times S^1$ are prime manifolds. The torus $\mathbb{T}^3$ is also prime.

Now, let us prove Theorem 1.1. Let M be a hyperbolic invariant set for a diffeomorphism f such that M is a 3-manifold. Then $f|M$ is a quasi-Anosov diffeomorphism [15]. Let us first assume that M is orientable. Theorem 4.1 implies that all attractors and repellers of $f|M$ are orientable. Then Theorems 4.2 and 4.3 imply that the basin of attraction/repulsion of each attractor/repeller is homeomorphic to a finitely punctured torus. Also, that f restricted to each basin is topologically equivalent to a DA-diffeomorphism. Let us first consider the simplest case of a non-Anosov quasi-Anosov diffeomorphism: one with just one attractor and one repeller. Let us furthermore suppose that the basin of attraction of the attractor is homeomorphic to a torus minus one point. If one takes a ball centered at that (repelling) point,

and cuts the pre-image of that ball under the DA-diffeomorphism, one obtains a fundamental domain, which is homeomorphic, under the conjugacy, to a fundamental domain D of $f|M$, met just once by the orbit of every point not in the attractor nor in the repeller of $f|M$. By connectedness, the basin of repulsion of the repeller must be also homeomorphic to a torus minus one (attracting) point A. And f restricted to this basin is also conjugated to another DA-diffeomorphism. The image of D under this second conjugacy consists of two spheres S^3, each bounding a ball containing A in its interior. So, the fact that D is a fundamental domain implies that M is the connected sum of two tori, just like in the Franks-Robinson example [3].

Let us suppose now that we have an attractor and a repeller, but that the basin of attraction is homeomorphic to a torus minus k points, with $k \geq 2$. One can assume that the k points are fixed under the DA-diffeomorphism by taking a sufficiently high iterate of the diffeomorphism. A connectedness argument shows that the basin of repulsion of the repeller is also homeomorphic to a torus minus k points. Now, the previous procedure shows that M is obtained by removing k 3-cells from each torus and matching the resulting boundaries, using an orientation reversing homeomorphism. Observe that this implies that

$$M = T_1 \# T_2 \# H_1 \# \ldots \# H_{k-1}$$

where T_1, T_2 are tori and the H_i are handles $S^2 \times S^1$. Indeed, instead of removing simultaneously the k 3-cells of each torus, one can only remove one 3 cell from each torus and glue along their boundaries by f, which reverses orientation. In this way one obtains the connected sum of two tori. The rest of the procedure consists in repeating $(k-1)$ times the operation: cutting two disjoint 3-cells of this connected sum and matching the resulting boundaries by a reversing orientation homeomorphism. This is the same as adding a handle.

Now, in fact there is nothing special in having just one attractor and one repeller. In case there are more attractors or repellers, one proceeds inductively as in the previous cases until one obtains a finite connected sum of tori and handles.

Let us also consider the case where the non-wandering set has a basic set Λ that is a periodic orbit, we may assume $\text{st}(\Lambda) = \text{st}(\Lambda_0)$, and that $\Lambda \to \Lambda_0$, where Λ_0 is a connected codimension-one attractor. Let p and q be associated boundary periodic points and $P_1, ..., P_k$ be the set of all periodic points with $\text{st}(P_i) = 1$ and

$$W^u(P_i) \cap \bigcup_{x \in W^u(p) - \{p\}} [x, \phi_{p,q}(x)]_{\emptyset}^s \neq \emptyset$$

for all $1 \le i \le k$. Let $x \in W^u(p) - \{p\}$ and

$$\theta_x : [x, \phi_{p,q}(x)]_{\emptyset}^s \to [0, 1]$$

be a homeomorphism. Denote P_i^x to be the point of intersection between $[x, \phi_{p,q}(x)]_{\emptyset}^s$ and $W^u(P_i)$. By reordering the points, if necessary, assume that $\theta_x(P_i^x) < \theta_x(P_j^x)$ for $i < j$. Then for any other $x' \in W^u(x')$ and $\theta_{x'}$ defined similarly we have $\theta_{x'}(P_i^{x'}) < \theta_{x'}(P_j^{x'})$ for $i < j$ since $W^u(P_i)$ is codimension-one.

For the rest of the construction we assume, with no loss of generality, that $p, q, P_1, ..., P_k$ are all fixed points. Define the set

$$D = \bigcup_{x \in W^u(p) - \{p\}} [x, \phi_{p,q}(x)]_{\emptyset}^s \cup W_{\emptyset}^s(p) \cup W_{\emptyset}^s(q) \cup W^s(P_1) \cup \cdots \cup W^s(P_k).$$

Following the construction in the proof of Theorem 1 of [20] we can extend the diffeomorphism $f|_D$ to a homeomorphism $\bar{f}$ on the compactification $\bar{D} = D \cup \alpha_1 \cup \cdots \alpha_{k+1}$ where each α_i is a repelling fixed point for $\bar{f}$. Fix $\epsilon > 0$ sufficiently small and let

$$B = \bigcup_{x \in W_{\epsilon}^u(p) - \{p\}} [x, \phi_{p,q}(x)]_{\emptyset}^s \cup W_{\emptyset}^s(p) \cup W_{\emptyset}^s(q) \cup \left(\bigcup_{i=1}^{k} W^s(P_i)\right) \cup \left(\bigcup_{i=1}^{k+1} \alpha_i\right).$$

Let $B_\epsilon(0)$ be the ball of size ϵ centered at the origin in $\mathbb{R}^{n-1}$. Then we can define a homeomorphism $F : B_\epsilon(0) \times [0, 1] \to B$ so that

- $F(0, 0) = p$,
- $F(0, 1) = q$,
- $F(0, \frac{2i}{2k+1}) = P_i$ for $1 \le i \le k$,
- $F(0, \frac{2i+1}{2k+1}) = \alpha_{i+1}$ for $0 \le i \le k$,
- $F(x, t) \in [F(x, 0), \phi_{p,q}(F(x, 0))]_{\emptyset}^s$, and
- $F(x, 0) \in W_{\epsilon}^u(p)$.

Furthermore, we can extend the unstable manifolds of the points in $p \cup P_1 \cup \cdots \cup P_k$ to a codimension-one fibration of $\bar{D}$ and extend the stable foliation to a fibration of $\bar{D}$ with one-dimensional fibers. Similar to Corollary 7.2 in [8], there exists a compact arc $a_{pq} \subset \bar{D}$ with no self intersections such that

$$a_{pq} = p \cup W_{\emptyset}^s(p) \cup \alpha_1 \cup W^s(P_1) \cup \alpha_2 \cup \cdots \cup P_k \cup \alpha_{k+1} \cup W_{\emptyset}^s(q) \cup q.$$

Let $\mathcal{P}$ be the set of saddle periodic points of stable index one that intersect $W^s(\Lambda)$. Following the above construction we compactify $W^s(\Lambda) \cup W^s(\mathcal{P})$ to a set $\overline{W^s(\Lambda)}$ and extend the diffeomorphism f on $W^s(\Lambda) \cup W^s(\mathcal{P})$ to a homeomorphism $\bar{f}$ of $\overline{W^s(\Lambda)}$. Where $\overline{W^s(\Lambda)} = W^s(\Lambda) \cup W^s(\mathcal{P}) \cup \mathcal{A}$ where $\mathcal{A}$ consists of a set of repelling periodic points for $\bar{f}$. The proof of Theorem 7.1 in [8] extends to $\overline{W^s(\Lambda)}$ to show that $\overline{W^s(\Lambda)}$ is homeomorphic to $\mathbb{T}^3$.

In the case where there is at least one non-orientable attractor or repeller, then M is non-orientable. Theorem 4.3 implies that the basin of attraction or repulsion of this attractor or repeller is homeomorphic to a torus quotiented by an involution minus k points. f is doubly covered by a DA-diffeomorphism in this set, and the procedure of removing cells and matching the corresponding boundaries follows as in the previous cases, whence one obtains that

$$M = \tilde{T}_1 \# \ldots \# \tilde{T}_n \# H_1 \# \ldots \# H_k$$

where the $\tilde{T}_j = \mathbb{T}^3 | \theta_j$ are tori quotiented by involutions θ_j (possibly the identity), and the H_i are handles.

6. AN EXAMPLE OF A QUASI-ANOSOV DIFFEOMORPHISM WITH A BASIC SET THAT IS A PERIODIC ORBIT

Both the example by Franks-Robinson [3], and the example by Medvedev-Zhuzhoma [18] are quasi-Anosov diffeomorphisms whose non-wandering set consists exclusively of one codimension-one expanding attractor and one shrinking repeller.

Let us construct an example of a quasi-Anosov diffeomorphism with a basic set consisting of a periodic orbit.

Let A be a linear hyperbolic diffeomorphism of a 3-torus T_1 having at least one fixed point, and such that the stable dimension is 1. Make a deformation around a fixed point, in order to obtain a DA-diffeomorphism with a repelling fixed point and a codimension-one expanding attractor. See details in Section 4. This new diffeomorphism h preserves the original stable foliation. Now, make a new deformation, also preserving the original stable foliation, such that the repelling fixed point turns into a saddle, and two repelling fixed points appear on its stable manifold, locally separated by the unstable manifold of the saddle point.

Like in Franks-Robinson example, cut two 3-balls B_2 and B_3 containing, respectively, the repelling fixed points of T_1. Now, take two 3-tori T_2 and T_3, and consider the dynamics of h^{-1} on each one of them. Cut two 3-balls B_2' and B_3', each containing the attracting fixed points of T_2 and T_3 respectively. Glue carefully T_1 and T_j along the boundary of B_j and B_j', $j = 2, 3$, by means of a orientation reversing homeomorphisms as described in section §5. One obtains an Axiom A diffeomorphism of a manifold which is the connected sum of T_1, T_2 and T_3: $T_1 \# T_2 \# T_3$. The non-wandering set of this diffeomorphism consists of one codimension-one expanding attractor, two codimension-one shrinking repellers, and a hyperbolic fixed point. Proceeding as in [3] one perturbs the diffeomorphism producing a twist in the regions where the surgery was performed. Since this perturbation is local, it does not affect the hyperbolic behavior of the non-wandering set.

In this way one obtains an Axiom A diffeomorphism satisfying the quasi-transversality condition, and with the above mentioned non-wandering set.

7. Partially hyperbolic quasi-Anosov diffeomorphisms

In this section, we study quasi-Anosov diffeomorphisms in the presence of partial hyperbolicity. A diffeomorphism f is called **partially hyperbolic** if there exists an invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ such that all unit vectors $v^\sigma \in E^\sigma_x$, $\sigma = s, c, u$ satisfy:

$$|Tf_x v^s| < |Tf_x v^c| < |Tf_x v^u| \quad \text{and} \quad |Tf_x v^s| < 1 < |Tf_x v^u|$$

It is a known fact that there are unique invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ that are everywhere tangent, respectively, to E^s and E^u (see, for instance [12]). However, E^c is not integrable in general. It is an open problem if it is integrable in the case dim $E^c = 1$. Here, we shall consider an a priori mild hypothesis: either E^{cs} or E^{cu} integrates to a (codimension-one) foliation.

Let us prove Theorem 1.3. Assume that E^{cu} integrates to a foliation $\mathcal{F}$. We may suppose that the manifold M is not the 3-torus, for otherwise the result is immediate. Then, M is not **irreducible** (that is, there is an embedded sphere S^2 not bounding any 3-dimensional ball). Indeed, irreducible manifolds are prime, and Theorem 1.1 implies that M is not prime unless it is the 3-torus. Now, Theorem C.2., p 45 of Roussarie [24] implies that a codimension-one foliation $\mathcal{F}$ of a manifold that is not irreducible has a compact leaf. This compact leaf T must be homeomorphic to a 2-torus, since the strong unstable foliation has no singularities and does not contain closed leaves.

Take $p \in M$ a periodic point in M, of period k, such that the stable manifold of p hits T. This implies that the set of leaves $f^{kn}(T)$ (f preserves the foliation) accumulates in p. Now, Haefliger [10] says that the set of points lying in a compact leaf is compact. In particular $p \in M$, accumulating point of compact leaves, must be in a compact leaf T_0. It follows that T_0 is f^k-invariant. Moreover, by the above argument, T_0 must be a 2-torus. Now, the local stable manifolds of T_0 form an open set U satisfying $f^k(\overline{U}) \subset U$. This implies that T_0 is a basic set that is an attractor, but this contradicts Theorem 1.2.(2).

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